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1996 J. Phys. A: Math. Gen. 29 2123

(http://iopscience.iop.org/0305-4470/29/9/023)

J. Phys. A: Math. Gen. **29** (1996) 2123–2126. Printed in the UK

A self-similar tiling of Euclidean space by two shapes in two sizes

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Received 24 October 1995, in final form 18 January 1996

Abstract. We present a set of tiles consisting of a tetrahedron and an octahedron in two sizes which admits a tile inflation yielding a non-periodic tiling of space.

In the mid-seventies Penrose (1978) discovered an aperiodic tiling of the plane by two shapes, the kite and the dart. It was remarked in Robinson (1975) that an equivalent tiling can be generated by two tiles which admit a *tile inflation*, i.e. each tile can be tiled by smaller tiles similar to the larger tiles. The two tiles considered by Robinson are the *golden triangles* with sidelengths $1, \tau$ and τ, and $\tau, 1$ and 1, where $\tau = \frac{1}{2}(1+\sqrt{5})$; see figure 1 for the tile inflation, where the magnification factor is τ. Three-dimensional analogues of these tilings have been given by Kramer (1982) (seven tiles), Mosseri and Sadoc (1982) (four tiles) and Danzer (1989) (four tiles). The purpose of this paper is to give a set of four tiles which tile space by tile inflation in a way which is closer to the 2D Penrose tiling than the examples above.

The set consists of four tiles, but we only have to describe two of them as the other two are just inflated copies of the first two (by a linear factor τ). Both tiles can already be found in Kramer's paper. The first is a tetrahedron denoted a, called the aetos (eagle), the second, denoted v, an octahedron called the laros (seagull); see figure 2 for unfoldings of these two cells. Apart from the feature that there are only two different shapes in the set, the two cells have the following attractive properties:

- the cells have the same symmetry group (there is a twofold rotation axis, and two perpendicular mirrorplanes),
- all faces of the cells are golden triangles (and all edge lengths are powers of τ).

We shall denote the set of tiles by $\mathcal{A}_1 := \{a, A, v, V\}$, where $A = \tau a$ and $V = \tau v$. The tile inflation rule for this set of polyhedra can be conveniently described by a substitution $\theta_1 : \mathcal{A}_1 \to \mathcal{A}_1^*$. The substitution is given by

$$\theta_1(a) = A \qquad \theta_1(v) = V \qquad \theta_1(A) = a^2vA^2 \qquad \theta_1(V) = ava^2vA^2VA^2VAV\,.$$

As yet, this is a formal way to indicate how the inflated tiles $\tau a, \ldots \tau V$ can be packed by copies of the tiles $a, \ldots, V$ (cf figure 3). However, as in Dekking (1982), this can be extended to a substitution over a larger alphabet which captures the complete geometry of the tiling. Due to symmetry breaking the larger alphabet should contain symbols corresponding

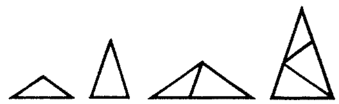

Figure 1. Left: the two golden triangles. Right: tile inflation of the golden triangles.

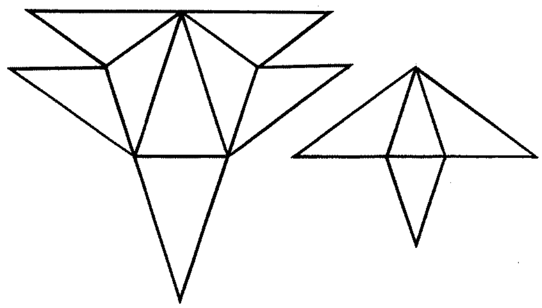

Figure 2. Unfoldings of the laros v (left) and the aetos a (right).

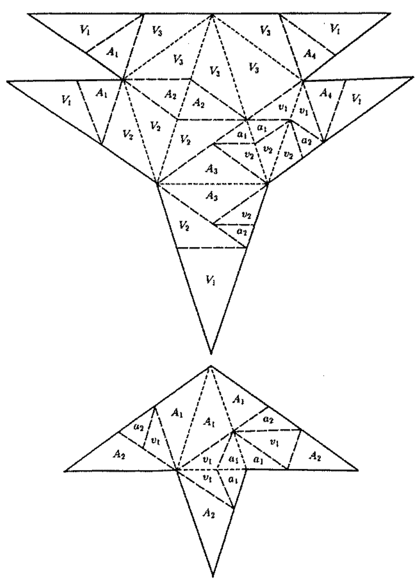

Figure 3. Tile inflation of τA and τV. Shown is how the faces of a_1, a_2, v_1, A_1 and A_2 cover the faces of τA_1 and how the faces of $a_1, a_2, v_1, v_2, A_1, A_2, A_3, A_4, V_1, V_2$ and V_3 cover the faces of τV. The tiles a_3 and A_5 are in the interior of τV.

to left-handed and right-handed versions of the tiles, and further tiles should be distinguished according to their icosahedral orientations (see Dekking (1995) for this construction for the Penrose tiling by golden triangles). Because of the two mirror versions of the tiles and the multitude of ways the cells can be packed in their inflated copies, there will be many $\{a, v, A, V\}$-tilings possible. As usual, the tiling of space is obtained by considering the patch of tiles described by $\theta_1^n(A)$ $n = 1, 2, \ldots$ which, when conveniently centred, covers larger and larger parts of space.

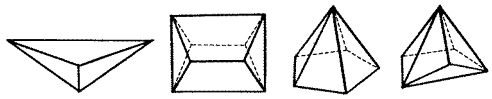

Figure 4. The A, H, S and Z tiles.

The substitution matrix corresponding to the inflation is given by

$$M = \begin{pmatrix} 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 2 & 5 \\ 0 & 1 & 0 & 3 \end{pmatrix}.$$

Clearly the matrix is primitive, as its third power has strictly positive entries only. Non-periodicity of the tiling of space by any tiling from this family follows as usual, by what is called the 'ratio test' in Senechal and Taylor (1990). A right eigenvector of the matrix M corresponding to the leading eigenvalue τ^3 gives the limiting relative numbers of each of the four types of tiles. Since these eigenvectors are proportional to $(2+\tau, 1, \tau^2, \tau)$, the irrational ratios of some of these frequencies prove non-periodicity of the tiling.

The $\{a, A, v, V\}$-tilings are closely related to the $\{A, H, S, Z\}$-tilings of Mosseri and Sadoc (1982). Here A is the same tetrahedron, H is an octahedron, S a pentagonal pyramid and Z a heptahedron (cf figure 4).

The tile inflation proposed by Mosseri and Sadoc can be described by a substitution $\theta_2 : \mathcal{A}_2 \to \mathcal{A}_2^*$, where $\mathcal{A}_2 = \{A, H, S, Z\}$. This substitution is given by

$$\theta_2(A) = SA^2, \quad \theta_2(H) = ZSA^2HSZ \qquad \theta_2(S) = A^2HSZ \qquad \theta_2(Z) = AHSZ\,.$$

As for the $\{a, v, A, V\}$-tiling, there is symmetry breaking and packing is not unique, so many tilings are possible. We claim that for any such tiling there is an $\{a, v, A, V\}$-tiling such that the two tilings are mutually locally derivable (as defined in Baake *et al* 1991). To prove this, we do not have to study vertex or edge configurations because we will derive the two tilings from each other by direct packing. Define the substitution $\sigma_{21} : \mathcal{A}_2 \to \mathcal{A}_1^*$ by

$$\sigma_{21}(A) = a^2vA^2 \qquad \sigma_{21}(H) = ava^2vA^2V \qquad \sigma_{21}(S) = A^2V \qquad \sigma_{21}(Z) = AV\,.$$

This substitution describes how τA, τH, τS and τZ can be packed by copies of a, v, A and V. To show that it is possible to pass from a specific Mosseri–Sadoc tiling (or rather its inflation by a linear factor τ) to an $\{a, v, A, V\}$-tiling, it suffices to check that

$$\sigma_{21} \circ \theta_2 = \theta_1 \circ \sigma_{21}$$

because then $\sigma_{21} \circ \theta_2^n = \theta_1^n \circ \sigma_{21}$ for all $n \geqslant 1$, and hence arbitrarily large patches from one tiling are transformed to the other by σ_{21} (or rather the more detailed extension of σ_{21}, which takes into account mirror versions and orientations). In fact we have, for example,

$$\sigma_{21} \circ \theta_2(Z) = \sigma_{21}(AHSZ) = a^2vA^2ava^2vA^2VA^2VAV$$

and

$$\theta_1 \circ \sigma_{21}(Z) = \theta_1(AV) = a^2vA^2ava^2vA^2VA^2VAV\,.$$

We leave it to the reader to check this relation for the other symbols A, H and S. To go the other way, we define $\sigma_{12} : \mathcal{A}_1 \to \mathcal{A}_2^*$ by

$$\sigma_{12}(a) = A \qquad \sigma_{12}(v) = HSZ \qquad \sigma_{12}(A) = SA^2 \qquad \sigma_{12}(V) = ZSA^2HSZA^2HSZAHSZ.$$

Again, it is evident that $\sigma_{12} \circ \theta_1 = \theta_2 \circ \sigma_{12}$, which implies that the corresponding Mosseri–Sadoc tiling can be locally derived from the $\{a, v, A, V\}$-tiling.

In this paper we hardly consider the question of inflation rules and matching rules. We conjecture that these exist for the $\{a, v, A, V\}$-tilings associated with the Mosseri–Sadoc tilings derived from the root lattice D_6 by Kramer and Papadopolos (1994).

Acknowledgment

I am grateful to the referees, whose remarks have led to a considerable improvement in the presentation of this paper.

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